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Spatially Covariant Gravity: perturbative analysis and field transformations

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Reference:
X. Gao, K. Chao (康超) and Z.-B. Yao, PRD, (2019)
X. Gao and Z.-B. Yao, [arXiv: 1806.02811]

The Lagrangians

Why this Lagrangian?

[Xian Gao, PRD, 2014]

$$\xrightarrow{\text{scalar}-\text{tensor}} \mathcal{L}\left(\phi, g_{\mu\nu}, {}^{(4)}R_{\mu\nu}; \nabla_\mu\right)$$

$$\xrightarrow{\text{ADM}} \mathcal{L}\left(\phi, N, h_{\mu\nu}, {}^{(3)}R_{\mu\nu}; \pounds_{\vec{n}}, D_\mu\right)$$

$$\xrightarrow{\text{unitary gauge}} \mathcal{L}\left(t, N, h_{ij}, {}^{(3)}R_{ij}; \pounds_{\vec{n}}, \nabla_i\right)$$

$$\xrightarrow{\text{special case}} \mathcal{L}\left(t, N, h_{ij}, {}^{(3)}R_{ij}, \pounds_{\vec{n}}h_{ij}; \nabla_i\right)$$

$$\xrightarrow{\text{extending}} \mathcal{L}\left(t, N, h_{ij}, {}^{(3)}R_{ij}, \pounds_{\vec{n}}N, \pounds_{\vec{n}}h_{ij}; \nabla_i\right)$$

Spatially Covariant Gravity with velocity of lapse function

[Xian Gao, Z.-B. Yao, arXiv:1806.02811]

To the XG theory, DoF is 3, but to the extended one, the DoF generally is 4.

Under what conditions it's reduced to 3 ?

Hamiltonian analysis

Results

Spatially covariant gravity with velocity of the lapse function:

$$S^{(\mathrm{u.g.})} = \int dt d^3x N \sqrt{h} \mathcal{L}\left(t, N, h_{ij}, F, K_{ij}, R_{ij}, \nabla_i\right) \quad F \equiv \pounds_{\vec{n}} N \quad K_{ij} \equiv \frac{1}{2} \pounds_{\vec{n}} h_{ij}$$

The degrees of freedom is 3 if the two conditions are satisfied:

$$\mathcal{D}\left(\vec{x}, \vec{y}\right) = 0, \qquad \text{(Degeneracy condition)}$$ → Degenerate kinetic matrix

$$\mathcal{F}\left(\vec{x}, \vec{y}\right) = 0, \qquad \text{(Consistency condition)}$$ → Existence secondary constraint

Degenerate Lagrangian is not sufficient to remove a DoF.

Hamiltonian analysis

The two conditions

$$0 = \mathcal{D}\left(\vec{x},\vec{y}\right) \equiv \frac{\delta^2 S}{\delta F\left(\vec{x}\right)\delta F\left(\vec{y}\right)} - \int d^3 z \int d^3 w \frac{\delta^2 S}{\delta F\left(\vec{x}\right)\delta K_{mn}\left(\vec{z}\right)} \mathcal{G}_{mn,pq}\left(\vec{z},\vec{w}\right) \frac{\delta^2 S}{\delta K_{pq}\left(\vec{w}\right)\delta F\left(\vec{y}\right)}$$

$$\begin{aligned}
0 &= \mathcal{F}\left(\vec{x},\vec{y}\right) \\
&\equiv \frac{1}{N\left(\vec{y}\right)} \frac{\delta^2 S}{\delta N\left(\vec{x}\right)\delta F\left(\vec{y}\right)} \\
&\quad + \int d^3 z \int d^3 w \left(\frac{1}{N\left(\vec{x}\right)} \frac{\delta^2 S}{\delta F\left(\vec{x}\right)\delta h_{mn}\left(\vec{z}\right)} - \frac{\delta}{\delta N\left(\vec{x}\right)} \left(\frac{1}{2N\left(\vec{z}\right)} \frac{\delta S}{\delta K_{mn}\left(\vec{z}\right)} \right)\right) \\
&\quad \times 2N\left(\vec{z}\right) \mathcal{G}_{mn,pq}\left(\vec{z},\vec{w}\right) \frac{1}{N\left(\vec{y}\right)} \frac{\delta^2 S}{\delta K_{pq}\left(\vec{w}\right)\delta F\left(\vec{y}\right)} \\
&\quad + \int d^3 z \int d^3 w \int d^3 z' \int d^3 w' \frac{1}{N\left(\vec{x}\right)} \frac{\delta^2 S}{\delta F\left(\vec{x}\right)\delta K_{p'q'}\left(\vec{w}'\right)} 2\mathcal{G}_{p'q',mn}\left(\vec{w}',\vec{z}\right) \\
&\quad \times N\left(\vec{z}\right) \frac{\delta^2 S}{\delta h_{mn}\left(\vec{z}\right)\delta K_{pq}\left(\vec{w}\right)} \mathcal{G}_{pq,m'n'}\left(\vec{w},\vec{z}'\right) \frac{1}{N\left(\vec{y}\right)} \frac{\delta^2 S}{\delta K_{m'n'}\left(\vec{z}'\right)\delta A\left(\vec{y}\right)} - \left(\vec{x} \leftrightarrow \vec{y}\right)
\end{aligned}$$

Concrete examples

Quadratic case

Quadratic Lagrangian (4 DoF) :

$$\mathcal{L}^{(\mathrm{quad})} = a_1 K + a_2 F + b_1 K_{ij} K^{ij} + b_2 K^2 + c_1 K F + c_2 F^2 + \mathcal{V}$$

where $a_1 \sim c_2$ are the functions of (t, N, X)

$$\mathcal{D}(\vec{x}, \vec{y}) = 0 \Rightarrow (b_1 + 3b_2)\, c_2 - \frac{3}{4} c_1^2 = 0$$

$$\mathcal{F}(\vec{x}, \vec{y}) = 0 \Rightarrow \frac{\partial a_2}{\partial X} - \frac{3}{2} \frac{c_1}{b_1 + 3b_2} \frac{\partial a_1}{\partial X} = 0 \quad \text{and} \quad \frac{\partial c_1}{\partial X} - \frac{c_1}{b_1 + 3b_2} \frac{\partial (b_1 + 3b_2)}{\partial X} = 0$$

Quadratic Lagrangian (3 DoF) :

$$\boxed{\mathcal{L}^{(\mathrm{quad})} = a_1 (K + \gamma F) + \alpha F + b_1 \left(K_{ij} K^{ij} - \frac{1}{3} K^2 \right) + \beta (K + \gamma F)^2 + \mathcal{V}}$$

Where a_1 and b_1 are the general functions of $(t, N, \partial_i N)$
while α and γ are the functions of (t, N) only.

Perturbative analysis

Linear perturbations

Quadratic Lagrangian (4 DoF) :

$$\mathcal{L}^{(\mathrm{quad})} = a_1 K + a_2 F + b_1 K_{ij} K^{ij} + b_2 K^2 + c_1 K F + c_2 F^2 + \mathcal{V}$$

The homogeneous and isotropic background:

$$ds^2 = -\left(e^{2A} - g^{ij} B_i B_j\right) dt^2 + 2a B_i dt dx^i + a^2 g_{ij} dx^i dx^j$$

Quadratic order action of scalar mode:

$$\begin{aligned} & S_2^S[\zeta, A, B] \\ = & \int dt \frac{d^3k}{(2\pi)^3} a^3 \Big(\mathcal{C}_{\dot\zeta^2} \dot\zeta^2 + \mathcal{C}_{\dot\zeta \dot A} \dot\zeta \dot A + \mathcal{C}_{\dot A^2} \dot A^2 - \mathcal{C}_{\dot\xi B} \dot\zeta \frac{\partial^2 B}{a} - \mathcal{C}_{\dot A B} \dot A \frac{\partial^2 B}{a} + \mathcal{C}_{B^2} \frac{\left(\partial^2 B\right)^2}{a^2} \\ & + \mathcal{C}_{\zeta^2} \zeta^2 + \mathcal{C}_{\dot\zeta A} \dot\zeta A + \mathcal{C}_{\zeta A} \zeta A + \mathcal{C}_{A^2} A^2 - \mathcal{C}_{AB} A \frac{\partial^2 B}{a} \Big) \end{aligned}$$

X. Gao, K. Chao and Z.-B. Yao, PRD, (2019)

Perturbative analysis

Linear perturbations

Solving the auxiliary field B by its EoM:

$$\mathcal{C}_{\dot{\xi}B}\dot{\zeta}+\mathcal{C}_{\dot{A}B}\dot{A}+\mathcal{C}_{AB}A-2\mathcal{C}_{B^2}\frac{\partial^2 B}{a}=0$$

$$\text{if}\quad \mathcal{C}_{B^2}=b_1+b_2=0$$

$$S_2^S\left[\zeta,A\right]$$
$$=\int dt\frac{d^3k}{(2\pi)^3}a^3\left(\mathcal{C}_{\dot{\zeta}^2}\dot{\zeta}^2+\mathcal{C}_{\dot{\zeta}\dot{A}}\dot{\zeta}\dot{A}+\mathcal{C}_{\dot{A}^2}\dot{A}^2+\mathcal{C}_{\zeta^2}\zeta^2+\mathcal{C}_{\dot{\zeta}A}\dot{\zeta}A+\mathcal{C}_{\zeta A}\zeta A+\mathcal{C}_{A^2}A^2\right)$$

$$\text{if}\quad \mathcal{C}_{B^2}=b_1+b_2\neq 0$$

$$S_2^S\left[\zeta,A\right]$$
$$=\int dt\frac{d^3k}{(2\pi)^3}a^3\left(\mathcal{D}_{\dot{\zeta}^2}\dot{\zeta}^2+\mathcal{D}_{\dot{\zeta}\dot{A}}\dot{\zeta}\dot{A}+\mathcal{D}_{\dot{A}^2}\dot{A}^2+\mathcal{D}_{\zeta^2}\zeta^2+\mathcal{D}_{\dot{\zeta}A}\dot{\zeta}A+\mathcal{D}_{\zeta A}\zeta A+\mathcal{D}_{A^2}A^2\right)$$

Perturbative analysis

Linear perturbations

Degenerate Hessian matrix:

$$\text{if} \quad \mathcal{C}_{B^2} = b_1 + b_2 = 0$$

$$\det \begin{pmatrix} \mathcal{C}_{\dot{\zeta}^2} & \frac{1}{2}\mathcal{C}_{\dot{\zeta}\dot{A}} \\ \frac{1}{2}\mathcal{C}_{\dot{\zeta}\dot{A}} & \mathcal{C}_{\dot{A}^2} \end{pmatrix} = 3\left(b_1 + 3b_2\right)c_2 - \frac{9}{4}c_1^2 = 0$$

$$\text{if} \quad \mathcal{C}_{B^2} = b_1 + b_2 \neq 0$$

$$\det \begin{pmatrix} \mathcal{D}_{\dot{\zeta}^2} & \frac{1}{2}\mathcal{D}_{\dot{\zeta}\dot{A}} \\ \frac{1}{2}\mathcal{D}_{\dot{\zeta}\dot{A}} & \mathcal{D}_{\dot{A}^2} \end{pmatrix} = \frac{2b_1}{3\left(b_1 + b_2\right)}\left(3\left(b_1 + 3b_2\right)c_2 - \frac{9}{4}c_1^2\right) = 0$$

Degeneracy condition:

$$\boxed{\left(b_1 + 3b_2\right)c_2 - \frac{3}{4}c_1^2 = 0 \Leftrightarrow \mathcal{D}\left(\vec{x}, \vec{y}\right) = 0}$$

Perturbative analysis

Linear perturbations

Degeneracy condition: $(b_1 + 3b_2)\, c_2 - \frac{3}{4} c_1^2 = 0 \Rightarrow \tilde{\zeta} \equiv \zeta + \frac{1}{3}\gamma A$

$$S_2^S\left[\tilde{\zeta}, A\right] = \int dt \frac{d^3k}{(2\pi)^3} a^3 \left(\mathcal{D}_{\dot{\zeta}^2}\dot{\tilde{\zeta}}^2 + \mathcal{C}_{\zeta^2}\tilde{\zeta}^2 + \mathcal{F}_{\dot{\zeta}A}\dot{\tilde{\zeta}}A + \mathcal{F}_{\zeta A}\tilde{\zeta}A + \mathcal{F}_{A^2}A^2\right)$$

We remove the extra scalar mode A only by the degeneracy condition at the linear perturbations level.

What about the consistency condition?

Perturbative analysis

Cubic order perturbations

Cubic order action of scalar mode

$$S_3^S\left[\tilde{\zeta},A,B\right] \supset \int dtd^3x\dot{A}\partial_iA\partial^iA\frac{a}{4}\Big[2\frac{\partial a_2}{\partial X}-\frac{3c_1}{b_1+3b_2}\frac{\partial a_1}{\partial X} +6\left(\frac{\partial c_1}{\partial X}-\frac{c_1}{b_1+3b_2}\frac{\partial\left(b_1+3b_2\right)}{\partial X}\right)H\Big]$$

$$\tilde{\zeta}=\zeta+\gamma A+\frac{1}{2}\left(\gamma+\frac{\partial\gamma}{\partial N}\right)A^2$$

$$\boxed{\frac{\partial a_2}{\partial X}-\frac{3}{2}\frac{c_1}{b_1+3b_2}\frac{\partial a_1}{\partial X}=0,\ \frac{\partial c_1}{\partial X}-\frac{c_1}{b_1+3b_2}\frac{\partial\left(b_1+3b_2\right)}{\partial X}=0\Rightarrow\mathcal{F}\left(\vec{x},\vec{y}\right)=0}$$

We need to impose the consistency condition at the cubic order perturbations in order to kill the extra scalar mode A.

There is no extra mode in arbitrarily high orders in perturbations.

Perturbative analysis

Inhomogeneous background

The inhomogeneous background:

$$ds^2 = -\left(\bar{N}\left(\vec{x}\right)\right)^2 dt^2 + \left(a\left(t\right)\right)^2 \bar{g}_{ij}\left(\vec{x}\right) dx^i dx^j$$

The quadratic order Lagrangian:

$$\begin{aligned}\mathcal{L}_2\left(A, B, \zeta\right) \supset\ & -c_1 a^2 \bar{\nabla}^2 B \dot{A} - 2c_2 a^2 \partial^i B \dot{A} \\ & + \bar{N} a \left(\frac{\partial a_2}{\partial X}\bar{N} + 3H\frac{\partial c_1}{\partial X}\right)\partial_i \bar{N} \partial^i A \dot{A} \\ & + 2c_2 a^3 \bar{N} \dot{A}^2 + 6c_1 a^3 \dot{A}\dot{\zeta} + 6\left(b_1 + 3b_2\right)\frac{a^3}{\bar{N}}\dot{\zeta}^2\end{aligned}$$

The quadratic order Lagrangian satisfied the degeneracy condtion:

$$\begin{aligned}\mathcal{L}_2\left(A, B, \tilde{\zeta}\right) \supset\ & \frac{1}{2}\dot{A}\partial_i A \partial^i \bar{N}\bar{N} a\left[\left(2\frac{\partial a_2}{\partial X} - \frac{3c_1}{b_1 + 3b_2}\frac{\partial a_1}{\partial X}\right)\bar{N}\right. \\ & \left. + 6\left(\frac{\partial c_1}{\partial X} - \frac{c_1}{b_1 + 3b_2}\frac{\partial\left(b_1 + 3b_2\right)}{\partial X}\right)H\right]\end{aligned}$$

Consistency condition

Field transformations

The Lagrangian

General quadratic Lagrangian (4 DoF):

$$\begin{aligned}\mathcal{L}^{(\mathrm{quad})} &= a_1 K + a_2 F + a_3 X^{ij} K_{ij} \\ &+ b_1 K_{ij} K^{ij} + b_2 K^2 + b_3 X^{ij} K_{ij} K + b_4 X^{ij} K_i^k K_{jk} + b_5 \left(X^{ij} K_{ij}\right)^2 \\ &+ c_1 K F + c_2 F^2 + c_3 X^{ij} K_{ij} F + \mathcal{V}\end{aligned}$$

$\mathcal{D}\left(\vec{x}, \vec{y}\right) = 0, \quad$ (Degeneracy condition)

$\mathcal{F}\left(\vec{x}, \vec{y}\right) = 0, \quad$ (Consistency condition)

General quadratic Lagrangian (3 DoF):

$$\begin{aligned}\mathcal{L}^{(\mathrm{quad})} &= b_1 \hat{K}_{ij} \hat{K}^{ij} + \hat{b}_2 \left(K + \gamma F\right)^2 + \hat{b}_3 X^{ij} \hat{K}_{ij} \left(K + \gamma F\right) \\ &+ b_4 X^{ij} \hat{K}_i^k \hat{K}_{jk} + b_5 \left(X^{ij} \hat{K}_{ij}\right)^2 + \mathcal{V}\end{aligned}$$

Field transformations

The Lagrangian

A special Lagrangian without F (3 DoF):

$$\begin{aligned}\mathcal{L}^{(\mathrm{ori})} &= b_1 \hat{K}_{ij}\hat{K}^{ij} + b_2 K^2 + b_3 X^{ij}\hat{K}_{ij}K \\ &\quad + b_4 X^{ij}\hat{K}_i^k \hat{K}_{jk} + b_5 \left(X^{ij}\hat{K}_{ij}\right)^2 + \mathcal{V}\end{aligned}$$

Field transformations:

$$h_{ij} \to e^{2\omega} h_{ij}, \qquad N \to e^{\lambda} N, \qquad N^i \to N^i$$

$$\Rightarrow X_{ij} \to e^{2\lambda}\left(1 + N\frac{\partial \lambda}{\partial N}\right)^2 X_{ij}, \qquad K_{ij} \to e^{2\omega-\lambda}\left(K_{ij} + h_{ij}\frac{\partial \omega}{\partial N}F\right)$$

$$\begin{aligned}\mathcal{L}^{(\mathrm{ori})} \to \mathcal{L}^{(\mathrm{quad})} &\equiv b_1 \hat{K}_{ij}\hat{K}^{ij} + \hat{b}_2 (K + \gamma F)^2 + \hat{b}_3 X^{ij}\hat{K}_{ij}(K + \gamma F) \\ &\quad + b_4 X^{ij}\hat{K}_i^k \hat{K}_{jk} + b_5 \left(X^{ij}\hat{K}_{ij}\right)^2 + \mathcal{V}\end{aligned}$$

Summary

- Spatially covariant gravity with velocity of the lapse function. (The most general scalar-tensor theory beyond Horndeski at this moment!)
- In homogeneous and isotropic background, Degeneracy condition is needed in linear perturbation and Consistency condition is needed in cubic order perturbation.
- In inhomogeneous background, Degeneracy condition and Consistency condition are needed in the in linear perturbation.
- Different theories can be related by field transformations.

Thank you!